

CHEKANOV-TYPE THEOREM FOR SPHERIZED COTANGENT BUNDLES

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ABSTRACT. We prove a Chekanov-type theorem for the spherization of the cotangent bundle ST^*B of a closed manifold B . It claims that for Legendrian submanifolds in ST^*B the property “to be given by a generating family quadratic at infinity” persists under Legendrian isotopies.

INTRODUCTION

The notion of a generating family is well known in symplectic and contact topology. The key role in many problems plays the theorem, claiming that the property of Lagrangian (correspondingly Legendrian) submanifolds in T^*B (corr. $J^1(B, \mathbb{R})$) to be given by generating family is an invariant under Hamiltonian (corr. contact) isotopy (see, for example, [Cha, Ch, EG, F, L-S, Vi]).

In this paper we prove a similar result for Legendrian submanifolds in spherizations of cotangent bundles:

Theorem 0.1. *Let B be a closed manifold, $E \rightarrow B$ be a smooth compact fibration. Let $\{L_t\}_{t \in [0,1]}$ be a legendrian isotopy of a compact Legendrian manifold $L_0 \subset ST^*B$. Suppose L_0 is given by a generating family $F: E \rightarrow \mathbb{R}$. Then there exists $N \in \mathbb{Z}_+$, such that L_t is given by a generating family $G_t: E \times \mathbb{R}^N \rightarrow \mathbb{R}$ of the form:*

$$G_t(e, q) = F(e) + Q(q) + f_t(e, q)$$

for a nondegenerate quadratic form Q on \mathbb{R}^N and compactly supported function f_t such that $f_0 = 0$.

A very closed statement to the theorem 0.1 is the Theorem 4.1.1 contained in [EG] (see also [F] for another close result) and the main aim of this paper is to fill a gap in the proof (see [P1]).

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1. GENERATING FAMILIES FOR LEGENDRIAN SUBMANIFOLDS IN ST^*

1.1. Contact structure on ST^ .* We recall standard notions from contact geometry [AG]. Let B be a smooth manifold. Denote by 0_B the zero section of the cotangent bundle T^*B . The spherisation ST^*B of the cotangent bundle T^*B is a quotient space under the natural free action of a multiplicative group of positive real numbers \mathbb{R}_+ on $T^*B \setminus 0_B$, a positive number a transform a pair (q, p) ($q \in B, p \in T_q^*B$) into the pair (q, ap) . The space ST^*B is a smooth manifold of the dimension $2\dim B - 1$ and it carries a natural cooriented contact structure ξ defined as follows. Consider the Liouville 1-form $\lambda = pdq$ on T^*B , it defines a cooriented (by λ itself) hyperplane field $\lambda = 0$ on $T^*B \setminus 0_B$. This cooriented hyperplane field is invariant with respect to the action of \mathbb{R}_+ and it is tangent to orbits of the action. Hence the projection of that hyperplane field to ST^*B is a cooriented hyperplane field on ST^*B which turns out to be a contact structure. We remark here, that there is no natural choice of a contact form on ST^*B .

1.2. Critical points and critical values of Legendrian manifolds in $J^1(B)$. Consider the space $J^1(B) = J^1(B, \mathbb{R}) = T^*B \times \mathbb{R}$ of 1-jets of functions on a closed manifold B . We say that a point of a Legendrian manifold $\Lambda \subset J^1(B)$ is a critical point of Λ if it projects to a zero section 0_B under the natural projection $J^1(B) \rightarrow T^*B$. We say that a number a is a critical value of $\Lambda \subset J^1(B)$ if a equals to the value of u -coordinate of a critical point of Λ .

1.3. Legendrian manifolds. The following notion generalise a notion of regular level set of a function on a manifold. Let us fix a number $c \in \mathbb{R}$. Suppose that Legendrian manifold $\Lambda \subset J^1(B)$ is transverse to $T^*B \times \{c\} \subset T^*B \times \mathbb{R} = J^1(B)$. Note that this implies that c is not a critical value of Λ . Consider a manifold $L^c = \Lambda \cap (T^*B \times \{c\})$. The intersection of L^c with $0_B \times \{c\}$ is empty and the restriction of the natural projection $(T^*B \setminus 0_B) \times \{c\} \rightarrow ST^*B$ to L^c is a legendrian immersion. We denote the image of L^c by Λ^c and we say in this situation that Λ^c is a c -reduction of Λ . Note that ST^*B is a contact reduction (in the sence of [EG, P1]) of $J^1(B) \setminus (0_B \times \mathbb{R})$ along any hypersurface given by equation $u = c$ in $J^1(B) \setminus (0_B \times \mathbb{R})$ and Λ^c is a reduction of Λ .

1.4. Legendrian manifolds and generating families. Let us recall firstly the definition of generating family in the space of 1-jets of functions. Let B be a manifold, consider the space $J^1(B) = T^*B \times \mathbb{R}$ of one jets of functions on B . The space $J^1(B)$ is a contact manifold with the canonical contact structure given by the form $du - \lambda$, where λ is (a lift of) the Liouville form on T^*B , u is the coordinate on the factor \mathbb{R} . A

smooth bundle $E \rightarrow B$ and a generic function $F: E \rightarrow \mathbb{R}$ generates an (immersed) Legendrian submanifold $\Lambda_F \subset J^1(B)$ as follows. Consider a fiber of the bundle $E \rightarrow B$, and a critical point of the restriction of the function F to this fiber. Denote by C_F the set consisting of all such points. For a sufficiently generic function F the set C_F is a smooth submanifold of the total space E . The genericity condition is that the equation $d_w F = 0$, where w is a local coordinate on a fiber (for a local trivialization of the bundle) of $E \rightarrow B$, satisfies the condition of the implicit function theorem. At any point z of C_F the differential $d_B F(z)$ of the function F along the base B is well defined. The rule $z \mapsto (z, d_B F(z), F(z))$ defines an immersion $l_F: C_F \rightarrow J^1(B)$ and its image is a Legendrian manifold Λ_F under definition. Function F is called a generating family for Λ_F . For a generating family F in a local trivialization $B \times W$ of E the manifold Λ_F is given by the formula:

$$\Lambda_F = \{(q, p, u) | \exists w_0 F_w(q, w_0) = 0, p = F_q(q, w_0), u = F(q, w_0)\},$$

where q, p are canonical coordinates on T^*B .

Now we define a Legendrian (immersed) submanifold Λ_F^c in the space ST^*B starting from a smooth bundle $\pi: E \rightarrow B$ and a generic function $F: E \rightarrow \mathbb{R}$. For a point $b \in B$ we denote by $C_F^c(b)$ the c -level of the restriction of the function F to C_F . Genericity conditions are the following – C_F is a manifold in a neighborhood of C_F^c and c is a regular value of F restricted to C_F . Consider a map $C_F^c \rightarrow ST^*B$: $z \mapsto (z, [d_B F(z)])$. This map is well defined since for $z \in C_F^c$ $d_B F(z) \neq 0$. Moreover, this map is a legendrian immersion and we denote its image by Λ_F^c . We will be interested in embedded Legendrian manifolds only.

We remark here that if F is a generating family for a manifold $\Lambda_F \subset J^1(B)$ then F is a generating family for a manifold $L_F \subset ST^*B$ if and only if Λ_F is transversal to the hypersurface $\{u = 0\}$ in $J^1(B)$.

1.5. Stabilization. Similarly to the $J^1(B)$ -case one can stabilize generating families for Legendrian manifolds in ST^*B . If $F: E \rightarrow \mathbb{R}$ is a generating family for a Legendrian $L = L_F \subset ST^*B$, then a function $G: E \times \mathbb{R}^K$ of the form

$$G(e, w) = G(e) + Q(w), e \in E, w \in \mathbb{R}^K$$

where Q is a non-degenerate quadratic form on \mathbb{R}^k is also a generating family for L .

2. LEGENDRIAN ISOTOPY LIFTING

Proof of Theorem 0.1. We follow the strategy of the proof of Theorem 4.1.1. in [EG]. The (generalized) Chekanov theorem [Ch, P] for $J^1(B)$ has almost the same statement as Theorem 0.1 – ST^*B is replaced

by $J^1(B)$. We can perturb the generating family F to a generating family \tilde{F} , such that \tilde{F} is a generating family for L_0 and at the same moment is a generating family for a closed submanifold in $J^1(B)$. Now Theorem 0.1 is a direct corollary of the corresponding $J^1(B)$ -result [P] in view of the following legendrian isotopy lifting lemma. \square

2.1. Legendrian isotopy lifting lemma. The following lemma claims contact isotopy lifting property formulated in [EG] for more general situations. Unfortunately the general statement is not true [P1] and, correspondingly, our proof use the specificity of the ST^* -situation.

Lemma 2.1. *Let B be a closed manifold and $c \in \mathbb{R}$. Consider a compact Legendrian manifold $\Lambda \subset J^1(B)$ such that its c -reduction Λ^c is well defined and Λ^c is an embedded manifold. Let $L_{t,t \in [0,1]}$ be a legendrian isotopy of $\Lambda^c = L_0$. Then there exists a legendrian isotopy $\Lambda_{t,t \in [0,1]}$, $\Lambda_0 = \Lambda$ such that for any $t \in [0,1]$ its c -reduction is defined and $\Lambda_t^c = L_t$.*

Proof. It is sufficient to prove the statement of the lemma for $c = 0$. Consider the legendrian isotopy L_t . By isotopy extension theorem there exists a contact flow $\varphi_{t \in [0,1]}$, such that $\varphi_t(L_0) = L_t$ for any $t \in [0,1]$. Any contact isotopy of ST^*B lifts to a (homogeneous) Hamiltonian flow on $T^*B \setminus 0_B$. More precisely – consider a Hamiltonian $H_t: T^*B \setminus 0_B \rightarrow \mathbb{R}$ such that $H_t(ap, q) = aH_t(p, q)$ for any positive number a (we will say that such a Hamiltonian is homogeneous). Then the flow of such a Hamiltonian function is well defined for all values of t and projects to a contact flow on ST^*B . Moreover, any contact flow on ST^*B could be given as a projection of a unique Hamiltonian flow above.

We take a homogeneous Hamiltonian H_t corresponding to the flow φ_t . Consider a function $K_t(p, q, u) = H_t(p, q)$ on $(T^*B \setminus 0_B) \times \mathbb{R} \subset J^1(B)$ as a contact Hamiltonian (see [AG]) with respect to the contact form $du - \lambda$. Any set $(T^*B \setminus 0_B) \times \{c\}$ is invariant under the flow generated by K_t and coincides on it with the flow of H_t under the forgetful identifications $T^*B \times \{c\} = T^*B$. Indeed, it follows from the explicit formula for the corresponding contact vector field: $\dot{u} = K - pK_p, \dot{p} = K_q - pK_u, \dot{q} = -K_p$ (see [AG]). The u -component of this contact vector field equals to zero since K is homogeneous. Hence the flow ψ_t generated by K satisfy $\psi_t(\Lambda \cap (T^*B \setminus 0_B)) = L_t$. In general it is impossible to extend ψ_t to a flow on the whole space $J^1(B)$ so we will change the function K_t . Let us fix an arbitrary smooth function $\tilde{H}_t: T^*B \rightarrow \mathbb{R}$ coinciding with H_t in a neighbourhood of infinity. Denote by P_t the function $P_t(p, q, u) = \tilde{H}_t(p, q)$ and by P_t^C ($C \in \mathbb{R}_+$) the function $P_t^C(p, q, u) = \frac{1}{C}P_t(Cp, q, u)$. We claim that for sufficiently big

C the legendrian isotopy of Λ generated by the contact flow Ψ_t^C of P_t^C satisfies the claim of lemma.

Let us fix a number a such that the absolute value of any critical value of Λ is bigger then $2a$. Denote by $X \subset \Lambda$ the subset formed by all points such that the absolute value of u -coordinate is at most a , by Y we denote the closure of its complement $\Lambda \setminus X$. The set X is a compact set and contained in $(T^*B \setminus 0_B) \times \mathbb{R}$. Take a neighborhood $U \subset T^*B$ of the zero section, the support of $P_t^C - K_t$ is contained in $U \times \mathbb{R}$ for sufficiently big C . Hence, for sufficiently big C , $\Psi_t^C(X) = \psi_t(X)$ for all $t \in [0, 1]$. It remains to show that for sufficiently big C the u -coordinate of any point in $\Psi_t^C(Y)$ could not be zero and hence zero reduction of $\Psi_t^C(\Lambda)$ is L_t . The coordinate u changes under the action of a contact Hamiltonian P^C according to the law: $\dot{u} = P_t^C - p \frac{\partial P_t^C}{\partial p}$. So it is sufficient to show that the speed of the u -coordinate uniformly tends to zero as C tends to infinity. The following general consideration finishes the proof.

Consider a smooth vector bundle V over a closed manifold M . We denote by $M(c)$ fiberwise multiplication by c . We say that a smooth function on V is positively homogeneous degree 1 at infinity if it coincides with a continuous positively homogeneous (i.e. $1/\alpha(M(\alpha))^*$ -invariant for any positive α) degree 1 function up to a sum with a compactly supported continuous function. Let v be a vector field tangent to fibers of V and coinciding with Euler vector field on each fiber of V . Consider an operator D sending a function g on V to $g - L_v g$. For a positively homogeneous function f the function Df is a compactly supported function. Denote by f^C the function $1/C(M(C))^*f$, i.e. for any $x \in V$ $f^C(x) = 1/C f(M(C)x)$.

Lemma 2.2. *For any smooth positively homogeneous of degree 1 at infinity function f , the C^0 -norm of $D(1/C(M(C))^*f)$ tends to zero while $C \rightarrow +\infty$.*

Proof. Indeed, $D(\frac{1}{C}(M(C))^*f) = \frac{1}{C}(M(C))^*D(f)$. Hence the C^0 -norm of $D(1/C(M(C))^*f)$ equals to the C^0 -norm of f divided by C . \square

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